Overdeterministic Fracture Analysis and Singular Value Decomposition

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INTRODUCTION

Using overdeterministic analysis to determine fracture parameters leads to unstable solutions in some cases because of ill-conditioning problems. The problem arises because some of the functions that constitute the overall field expressions are similar. In these cases, the solution is best obtained using singular value decomposition, which solves for the coefficients effectively even when ill-conditioning is present, and quantifies the condition of the matrix. The method is explained and its use is demonstrated with two sets of results for interfacial cracks in thermally loaded bimaterials.

DEVELOPMENT

In a homogeneous, linear elastic, isotropic material, the horizontal component of displacement in a cracked body under mode I loading can be expressed as an infinite series: [1]

$$\begin{aligned} L(r,\theta) &= u_x = \\ &\frac{1}{E}\sum_{n=0}^{\infty} C_{2n}\frac{r^{n+1/2}}{n+1/2}\{(1-\nu)\cos[(n+1/2)\theta] \\ &- (1+\nu)(n+1/2)\sin(\theta)\sin[(n+1/2)\theta]\} \\ &+ C_{2n+1}\frac{r^{n+1}}{n+1}\{2\cos[(n+1)\theta] \\ &- (1+\nu)(n+1)\sin(n\theta)\sin(\theta)\} \end{aligned} \quad (1)$$

Here E is Young's modulus, u_x is the horizontal displacement component, r and θ are polar coordinates, and ν is Poisson's ratio. Methods such as the local collocation method or the boundary collocation method require the truncation of this expression to a finite size. The researcher then measures the left-hand side of eq. (1) at m discrete points, yielding an overdetermined system that can be solved for the n unknowns by assuming random measurement errors and formulating this least-squares using linear algebra. [2]

One possible difficulty lies with the experimental data. A moiré method, for example, measures overall displacements, which, in addition to the crack-induced displacement component of eq. (1), includes a displacement component related to translation and rotation of the body about some point: [1]

$$M(r,\theta) = Pr\cos(\theta) + Qr\sin(\theta) + R \quad . (2)$$

The parameters P, Q, and R must also be treated as unknowns so the actual field equation used in the overdeterministic method is a combination of eqs. (1) and (2): [1]

$$N(r,\theta) = L(r,\theta) + M(r,\theta) \quad (3)$$

Trying to use linear algebra to solve this equation will give poor results unless the researcher realizes that the P term in eq. (2) and the C_1 term in eq. (1) are linearly dependent terms. The two terms can be combined so that a nonsingular matrix is formed. Such a linear dependence will be manifested as a singular matrix. A similar situation occurs with the Q term when u_y fields are measured.

In the case of an interfacial crack lying between two materials, a field expression similar to eq. (1) exists, but is much more complicated because the solution for the bimaterial problem has complex eigenvalues. [3] The imaginary part of these eigenvalues is denoted ϵ and is called the *bimaterial parameter*. The value for the bimaterial parameter depends on the stress state (plane stress or plane strain) and the material properties on each side of the crack. The crack induced fields can be expressed as: [4]

$$\begin{aligned} L(r,\theta) &= \frac{1}{2\mu_1}[a_{0r}r^{1/2}(f_{0r})_1 - a_{0j}r^{1/2}(f_{0j})_1 + \\ &\quad b_{0r}r(g_{0r})_1 - b_{0j}r(g_{0j})_1] \quad y\geq 0 \\ &= \frac{1}{2\mu_2}[a_{0r}r^{1/2}(f_{0r})_2 - a_{0j}r^{1/2}(f_{0j})_2 + \\ &\quad b_{0r}r(g_{0r})_2 - b_{0j}r(g_{0j})_2] \quad y<0 \end{aligned} \quad (4)$$

Here μ_1 and μ_2 are the two shear moduli. The actual expression is an infinite series but for simplification has been truncated here. The terms with the $r^{1/2}$ dependencies are associated with the complex stress intensity factor, and those linear in r with constant stress terms. The functions f and g shown in eq. (4) depend on the polar coordinate θ, the material properties, and ϵ.

One might expect that the combination of rigid body motion and crack induced displacement components would again give two linearly dependent terms, resulting in a singular matrix. However, due to the presence of the bimaterial parameter, these terms are *almost*, but not quite, linearly dependent. In the limit as ϵ goes to zero, the matrix becomes singular, and the condition number approaches infinity. [4] For most combinations of materials, however, ϵ is small (on the order of 0.01 - 0.10) resulting in a nonsingular but ill-conditioned matrix.

Ill-conditioning of the matrix for interfacial fracture problems can then come from two sources: (i) the presence of similar (but not identical) terms when combining eqs. (2) and (4) and (ii) the errors introduced by lack of computational precision. Ill-conditioning caused by precision errors can sometimes be improved by the use of double precision numbers, but the similar terms in the field expression are an inherent feature. Worse yet, a researcher unaware of the ill-conditioning can still obtain a solution, and it may appear to be correct. The unstable solution is manifested by sporadic changes in the values for the determined parameters as the size of the truncated series is changed. [5]

A resolution is to use a more robust solution method. Linear algebra methods such as Gauss-Jordan elimination, Gaussian elimination with partial pivoting, and the Gauss-Seidel iterative technique have frequently been used with homogenous cracked bodies. Regardless of which of these is used, the design matrix, [*f*], is formed from the functions in the combined expression and depends on the measured locations of the experimental data points. The vector of displacement component measurements, {*N*}, and the vector of unknown parameters, {*C*}, relate to [*f*] through:

$$\underset{m\times 1}{\{N\}} = \underset{m\times n}{[f]}\ \underset{n\times 1}{\{C\}} \tag{5}$$

The conventional matrix approach proceeds by premultiplication with the transpose of [*f*] to produce a square matrix, [*a*], which is then solved using one of these techniques [2]:

$$\begin{aligned}
&[f]^T\{N\} = [f]^T[f]\{C\}\\
&\text{Let}\ \{d\} = [f]^T\{N\},\quad [a] = [f]^T[f]\\
&\underset{n\times 1}{\{d\}} = \underset{n\times n}{[a]}\ \underset{n\times 1}{\{C\}}
\end{aligned} \tag{6}$$

Attempts to apply this algorithm to the interfacial problems results in unstable solutions. A preferred technique is to use the Singular Value Decomposition (SVD) technique to decompose the design matrix into three separate parts: [5]

$$\{N\} = [f]\{C\} = \underset{m\times n}{[U]}\ \underset{n\times n}{[w]}\ \underset{n\times n}{[V]^T}\{C\} \tag{7}$$

The matrix [*w*] is a diagonal matrix with all nonnegative diagonal elements. These vectors and matrices and their various parts have interpretations from linear algebra that can be applied to the overdeterministic interfacial fracture problem. For example, since [*w*] is diagonal, the inverse is easily found, and linear algebra operations give the unknown coefficient vector as: [5]

$$\{C\} = [V]\cdot[diag(1/w_j)]\cdot(U^T\cdot\{N\}) \tag{8}$$

Also, the condition number of [*f*] can be defined as the ratio of the maximum to the minimum value of the diagonal elements of [*w*]. Large condition numbers indicate ill-conditioning. When the reciprocal of the condition number approaches the computational precision, the presence of ill-conditioning will significantly affect solution accuracy. If this occurs, the adverse effects can be minimized by setting the terms $1/w_j$ in eq. (8) equal to zero for all sufficiently small w_j. In this work, $1/w_j$ was replaced by zero if it was less than 1×10^{-12} of the maximum diagonal value. [5]

The effect of ill-conditioning is to produce an infinite set of solutions that all approximately solve the linear equation $\{N\} = [f]\{C\}$. Zeroing these diagonal elements selects from this set the solution ~~the one~~ that minimizes the residual $R = |\{N\} - [f]\{C\}|$. The result is that the SVD method with correction is often better than both direct methods and uncorrected SVD methods, as is shown by the experimental results below. [5]

RESULTS

Singular Value Decomposition has been used in two examples provided here. In both cases, a specimen had a heat source applied to its upper surface and a cooling plate applied to its lower surface, so that heat transfer occurred vertically and the temperature field was one-dimensional. We used implanted thermocouples to determine the temperatures and then used this data with finite element models to determine the *J* integral and the magnitudes of the complex stress intensity factors. Figures 1 and 2 show the two specimens. The first specimen was made from aluminum and copper, and the second was made from steel and a thermocouple cement. The material properties for the four materials are given in Table 1. The bimaterial parameters for these two combinations were 0.026 and 0.113, respectively, for the aluminum-copper and steel-cement material combinations.

Table1 Material properties for the bimaterial components

Material	Young's Modulus [GPa]	Poisson's ratio
aluminum	71.7	0.34
copper	120.0	0.33
steel	218.0	0.29
thermocouple cement	3.2	0.30

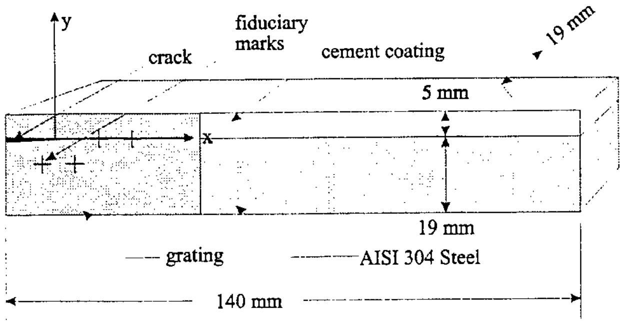

Figure 1 Geometry of the steel-thermocouple cement specimen

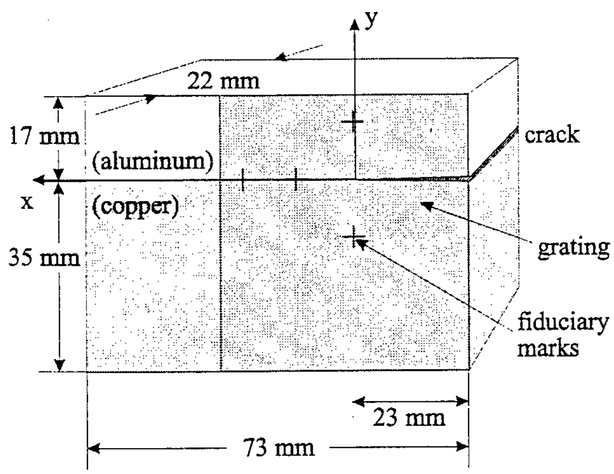

Figure 2 Geometry of aluminum-copper specimen

Identical thermal boundary conditions were applied to analogous specimens with moiré interferometry gratings. The resulting u_x displacement components were recorded as fringe patterns and used with the local collocation method. The real and imaginary parts of a_0 in eq. (4) are a_{0i} and a_{0r}, and are related to the real and imaginary parts of the complex stress intensity factor. Figure 3 shows the calculated values for a_{0i} and a_{0r} for the aluminum-copper specimen, and Figure 4 shows the results for the steel-cement specimen. The other curves shown in the figures are obtained using the SVD method without correcting for ill-conditioning (in test cases, the use of Gaussian elimination gave identical results to the uncorrected SVD method). Table 2 gives the magnitudes of the complex stress intensity factors calculated using the local collocation method with the SVD method and the related finite element calculations.

Table2 Magnitude of complex stress intensity factors for the two experiments [MPa $m^{1/2}$]

Material Pair	SVD-Local Collocation Method	Finite Element Results
aluminum-copper	0.613	0.601
steel-thermocouple cement	12.1	11.1

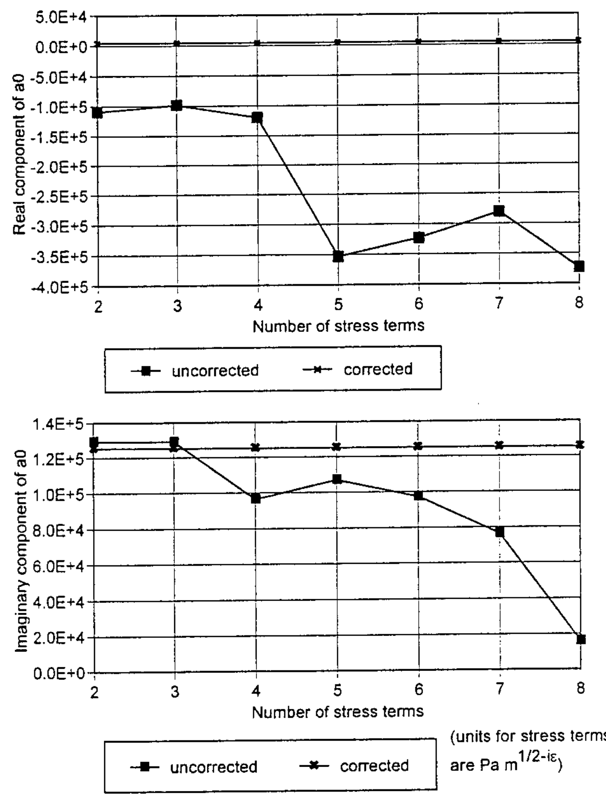

Figure 3 Comparison of results for aluminum-copper specimen

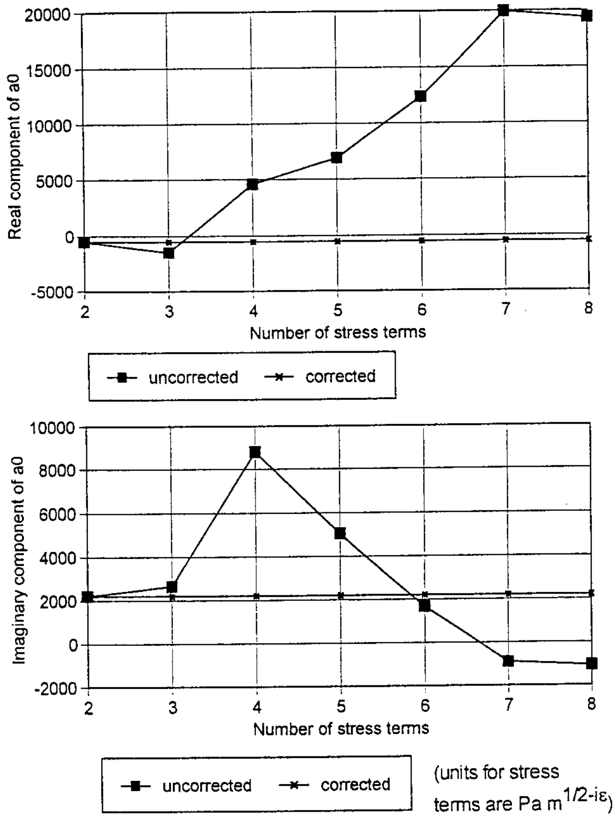

Figure 4 Comparison of results for the steel-thermocouple cement specimen

CONCLUSIONS

Ill-conditioning caused by a weak dependence of terms in full field interfacial crack data can cause instabilities in the solution of overdeterministic methods. An alternative solution technique, the singular value decomposition method, is available, and gives stable results that are unobtainable with direct methods. Applying this technique to experimental data for two different bimaterial combinations gave results that agreed with computational results.

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